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Ramsey numbers for a disjoint union of some graphs

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ABSTRACT

We give the Ramsey number for a disjoint union of some G-good graphs versus a graph G generalizing the results of Stahl [S. Stahl, On the Ramsey number $r(F, K_m)$ where F is a forest, Canad. J. Math. 27 (1975) 585–589] and Baskoro et al. [E.T. Baskoro, Hasmawati, H. Assiyatun, Note. The Ramsey number for disjoint unions of trees, Discrete Math. 306 (2006) 3297–3301].

1. Introduction

Let G, H, F be simple graphs with at least two vertices. The Ramsey number $R(G, H)$ is the smallest integer n such that every graph F of order n contains a subgraph isomorphic to G or $\overline{F}$ contains a subgraph isomorphic to H, where $\overline{F}$ is the complement of F.

For graphs G, H such that H is a subgraph of G, let us define $G - H$ as the graph obtained from G by deleting the vertices of H and all edges incident to them.

The graph H is G-good if $R(H, G) = (\chi(G) - 1)(|V(H)| - 1) + s(G)$, where $s(G)$ is the chromatic surplus of G, i.e. the minimum cardinality of colour classes over all chromatic colourings of $V(G)$. Let T_i be a tree of order i.

Chvátal [4] proved that $R(T_n, K_m) = (m - 1)(n - 1) + 1$, where T_n is a tree of order n, K_m is a complete graph of order m, with $n, m > 1$. Thus trees are K_m-good graphs.

Let F be a graph, $c(F)$ be the order of the largest component of F and $k_i(F)$ be the number of components of order i in F.

Stahl [7] extended the result of Chvátal for the family of disjoint trees and proved the following theorem.

Theorem 1 (*Stahl [7]*). *If F is an arbitrary forest then*

$$R(F, K_m) = \max_{1 \le j \le c(F)} \left\{ (j - 1)(m - 2) + \sum_{i=j}^{c(F)} i k_i(F) \right\}.$$

Baskoro et al. [1] proved the equivalent form of the formula of Stahl for special cases of forests. Their result is cited below.

Theorem 2 (*Baskoro et al. [1]*). *Let $n_i \ge n_{i+1}$ for $i = 1, 2, \ldots, k - 1$ and $m \ge 2$. If $n_i \ge (n_i - n_{i+1})(m - 1)$ for any i, then*

$$R\left(\bigcup_{i=1}^{k} T_{n_i}, K_m\right) = R(T_{n_k}, K_m) + \sum_{i=1}^{k-1} n_i.$$

We prove a generalization of both of the above results, where instead of K_m we consider a graph G with $s(G) = 1$, and instead of a family of trees we consider a family of some graphs consisting of G-good components. Further extension of the results to a more general class of graphs with G-good components for $s(G) \ge 1$ is presented in [2].

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2. Results

First we prove the following lemma.

Lemma A. *Let G be a graph with $\chi(G) = m$ and $s(G) = 1$. If B is a graph with t G-good components, each of them with n vertices, then*

$$R(B, G) = (n-1)(m-2) + nt.$$

Proof. The graph $H = K_{nt-1} \cup (m-2)K_{n-1}$ does not contain B. Moreover, $\chi(\overline{H}) \le m-1$, which means that $\overline{H}$ does not contain G. So we get $R(B, G) \ge (n-1)(m-2) + nt$. The reverse inequality is proved by induction on t. For $t = 1$ the assertion holds since the lemma is reduced to the definition of a G-good graph with $s(G) = 1$.

So assume the assertion of the lemma for all graphs with $t-1$ ($t > 1$) components each of which is a G-good graph on n vertices with $s(G) = 1$.

Let H be a graph on $(n-1)(m-2) + nt$ vertices such that $\overline{H}$ does not contain G. We will prove that H contains B. Let C be a component of B. Since $|V(H)| = (n-1)(m-2) + nt > (n-1)(m-2) + n$ we conclude that H contains C. Note that $|V(H) - V(C)| = (n-1)(m-2) + n(t-1)$. In view of the inductive hypothesis, we get that $H - C$ contains $B - C$. So H contains B. $\square$

By Lemma A and the idea of the proof for a forest versus a complete graph presented in [7] we get the following more general theorem.

Theorem 3. *Let G be a graph with $\chi(G) = m$ and $s(G) = 1$. If B is a graph with G-good components then*

$$R(B, G) = \max_{1 \le j \le c(B)} \left\{ (j-1)(m-2) + \sum_{i=j}^{c(B)} ik_i(B) \right\}.$$

Proof. Let B_j be the subgraph of B consisting of all the components with at least j vertices, where $1 \le j \le c(B)$. Evidently, $B_j - B_{j+1}$ consists of $k_j(B)$ components with exactly j vertices. Moreover, B_{j+1} is the subgraph of B_j. Suppose that the maximum is achieved for $j = j_0$. Set $p_0 = \sum_{i=j_0}^{c(B)} ik_i(B)$. Let $r = (j_0-1)(m-2) + p_0$. Note that the graph $H = K_{p_0-1} \cup (m-2)K_{j_0-1}$ does not contain any subgraph B_{j_0}. So H does not contain the graph B. Moreover, since $\chi(G) = m$ and $\chi(\overline{H}) \le m-1$, the complement of H does not contain G. Thus $R(B, G) \ge r$. To prove the reverse inequality suppose that the complement of a graph H of order r does not contain G. We shall show that H contains B. Let us assume that $c(B) = n$. Note that $r \ge (n-1)(m-2) + nk_n(B)$. Thus, by Lemma A, we have that H contains B_n. Now we use descending induction to show that H contains B_j for each $j \ge 1$. Let us state the inductive hypothesis: H contains B_{j+1} for some $1 \le j < n$. Note that B_{j+1} has $\sum_{i=j+1}^{n} ik_i(B)$ vertices. Thus $H - B_{j+1}$ has $r - \sum_{i=j+1}^{n} ik_i(B)$ vertices. By the definition of r we get $r \ge (j-1)(m-2) + \sum_{i=j}^{n} ik_i(B)$. Hence, $r - \sum_{i=j+1}^{n} ik_i(B) \ge (j-1)(m-2) + jk_j(B) = R(B_j - B_{j+1}, G)$. The above equality follows by Lemma A. So the graph $H - B_{j+1}$ contains $B_j - B_{j+1}$ and therefore H contains B_j. By induction, H contains B_1. $\square$

Similarly the result of Baskoro et al. can be extended for graphs G and B such that $s(G) = 1$ and B has G-good components. The theorem below presents a generalization.

Theorem 4. *Let G be a graph with $\chi(G) = m$ and $s(G) = 1$. Let $B = \bigcup_{i=1}^{k} B_{n_i}$, where B_{n_i} is a connected G-good graph of order n_i ($1 \le i \le k$). Let for $i = 1, 2, \dots, k-1$, $n_i \ge n_{i+1}$ and $p(n_i) = 1$ if $n_i \ge (n_i - n_{i+1})(m-1)$, and $p(n_i) = 0$ in the opposite case. If $p(n_i) = 0$ for each $j \le i \le k-1$ and $p(n_i) = 1$ for $1 \le i \le j-1$, where j is an integer $1 \le j \le k$, then*

$$R(B, G) = (m-1)(n_k - 1) + \sum_{i=1}^{k-1} n_i p(n_i) + (m-1) \sum_{i=1}^{k-1} (n_i - n_{i+1})(1 - p(n_i)) + 1.$$

Proof. Let $t = (m-1)(n_k-1) + \sum_{i=1}^{k-1} n_i p(n_i) + (m-1)\sum_{i=1}^{k-1}(n_i - n_{i+1})(1-p(n_i)) + 1$.

First we prove the inequality $R(B, G) \le t$ by induction on k. For $k = 1$ the result is immediate.

Let us assume the inductive hypothesis for all graphs consisting of s components (each being a G-good graph), where $1 \le s < k$:

$$R\left(\bigcup_{i=1}^{s} B_{n_i}, G\right) = (m-1)(n_s - 1) + \sum_{i=1}^{s-1} n_i p(n_i) + (m-1)\sum_{i=1}^{s-1}(n_i - n_{i+1})(1 - p(n_i)) + 1.$$

Let us take an arbitrary graph H of order t. Suppose that $\overline{H}$ does not contain G as a subgraph. Note that

$$t = (m-1)(n_{k-1} - 1) + 1 + \sum_{i=1}^{k-2} n_i p(n_i) + (m-1)\sum_{i=1}^{k-2}(n_i - n_{i+1})(1 - p(n_i)) + t_0,$$

where

$$t_0 = -(m-1)(n_{k-1} - n_k) + n_{k-1}p(n_{k-1}) + (m-1)(n_{k-1} - n_k)(1 - p(n_{k-1})).$$

Note that $t_0 \geq 0$. So, by the inductive hypothesis, H contains $F = \bigcup_{i=1}^{k-1} B_{n_i}$.

Note that $t - \sum_{i=1}^{k-1} n_i \geq (m-1)(n_k - 1) + 1$. So the graph $H - F$ contains B_{n_k}. Hence, H contains $B = \bigcup_{i=1}^{k} B_{n_i}$.

To prove the inequality $R(B, G) \geq t$ let us define a graph H of order $t - 1$ such that H does not contain B as a subgraph and $\overline{H}$ does not contain G as a subgraph.

We first note that $t - 1 = (m-2)x + y$, where $x = n_k - 1 + \sum_{i=1}^{k-1}(n_i - n_{i+1})(1 - p(n_i))$ and $y = \sum_{i=1}^{k} n_i - 1 - \sum_{i=1}^{k-1} n_{i+1}(1 - p(n_i))$. Moreover, by the assumption of the theorem, we have $x = n_j - 1$ and $y = \sum_{i=1}^{j} n_i - 1$.

Thus, let us consider the graph $H = (m-2)K_x \cup K_y$. Note that $B_{n_i} \not\subset K_x$ for $i \leq j$.

If $B \subset H$, then $\bigcup_{i=1}^{j} B_{n_i} \subset K_y$. But this is impossible because $\sum_{i=1}^{j} n_i > y$. Finally, $G \not\subset \overline{H}$ (since $\chi(G) - 1 = \chi(\overline{H})$). □

3. Remarks

Finally, let us consider some examples of the graphs consisting of G-good components, where $s(G) = 1$. Let $B = T_7 \cup T_5 \cup T_2$ and $F = T_5 \cup T_2 \cup T_2$, where T_n is a tree of order n. Each component of the graphs B and F is K_3-good. Note that $p(n_1) = 1$ and $p(n_2) = 0$ for the graph B. Thus the condition of Bascoro et al. does not hold for B. The assumption of Theorem 4 holds and we can apply the respective formula for counting $R(B, K_3) = 16$ (we can use Theorem 3, as well). Then note that $p(n_1) = 0$ and $p(n_2) = 1$ for the graph F. Thus the assumption of Theorem 4 does not hold for F and K_3. Evidently, the condition of Bascoro et al. does not hold as well. By Theorem 3, we have $R(B, K_3) = 10$. Note that the right hand side of the formula in Theorem 4 is equal to 11 for this pair of graphs. Similarly, the components of the graphs $B = C_{12} \cup C_5 \cup C_5$ and $F = C_{23} \cup C_{12} \cup C_5 \cup C_5$ are C_5-good, where C_n is a cycle of order n. So, by Theorem 3, we get $R(B, C_5) = 26$ and $R(F, C_5) = 48$ and the right hand of the formula in Theorem 4 is 28 and 50, respectively. In fact, the graphs B and F do not satisfy the assumption of Theorem 4. Evidently, the assumption holds for all graphs B with G-good components, where $\chi(G) = 2$ and $s(G) = 1$.

Problem. We can formulate the following problem. Characterize the sequences of orders of the components of a graph B for which the formula of Theorem 4 holds. Evidently, if each component of a graph B is G-good and $s(G) = 1$, then the right hand of the formula in Theorem 4 is the upper bound for $R(B, G)$ (see the first part of the proof of Theorem 4).

One can find some other examples of G-good graphs with $s(G) = 1$ in [3,5,6].

An attractive collection of applications of various branches of Ramsey theory in algebra, geometry and point-set topology is presented in B. Bollobás, Graph Theory, Springer GTM63, 1979.

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